ON THE PROOF OF UNIVERSALITY FOR ORTHOGONAL AND SYMPLECTIC ENSEMBLES IN RANDOM MATRIX THEORY

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Abstract. We give a streamlined proof of a quantitative version of a result from [DG1] which is crucial for the proof of universality in the bulk [DG1] and also at the edge [DG2] for orthogonal and symplectic ensembles of random matrices. As a byproduct, this result gives asymptotic information on a certain ratio of the $\beta = 1, 2, 4$ partition functions for log gases.

For $m \geq 2$, let

$$(1)\qquad \begin{aligned} h(x) &= \sum_{k=0}^{m-1} \beta_k x^{2k} \\ \beta_k &= 2\frac{(2m)(2m-2)\cdots(2m-2k)}{(2m-1)(2m-3)\cdots(2m-2k-1)}, \qquad 0 \leq k \leq m-1. \end{aligned}$$

For odd q set

$$(2)\qquad I(q) \equiv \frac{2}{\pi}\sin\frac{q\pi}{2}\int_{-1}^{1}\frac{\cos(q\arcsin x)}{h(x)(1-x^2)}\,dx = \frac{4}{\pi}\int_0^{\pi/2}\frac{\sin qs}{\sin s\ h(\cos s)}ds$$

and

$$(3)\qquad Q(q) \equiv I(q) + \frac{1}{2m}.$$

For $n \equiv 2m-1$, define the $(m-1)\times(m-1)$ matrix

$$(4)\qquad T^{[m-1]} = I - \frac{(m!)^2}{m(2m)!}Q^{[m-1]}B^{[m-1]} \equiv I - K^{[m-1]}$$

where

$$Q^{[m-1]}_{ij} = Q(n-2i+2j), \qquad B^{[m-1]}_{ij} = 2m\binom{n}{j-i}, \qquad 1 \leq i,j \leq m-1.$$

Here $\binom{n}{k} \equiv 0$ for $k < 0$.

In [DG1, Theorem 2.6], the authors prove the following result: for $m \geq 2$,

$$(5)\qquad \det T^{[m-1]} \neq 0$$

(see Remark 2 after the proof of Theorem 1 below). Note that in the notation of [DG1], $T^{[m-1]} = T_{m-1}$.

In this paper we will give a streamlined proof of the following quantitative version of (5).

Theorem 1. *For $m \geq 2$,*

$$(6)\qquad \det T^{[m-1]} \geq 0.0865.$$

Equation (5) plays a crucial role in proving universality in the bulk [DG1], and also at the edge [DG2], for orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) random matrix ensembles for a class of weights $w(x) = e^{-V(x)}$ where $V(x)$ is a polynomial $V(x) = \kappa_{2m}x^{2m} + \cdots$, $\kappa_{2m} > 0$. (Here m is the same integer as in (5), (6).) The situation is as follows. In [DG1, DG2], and also in [DGKV], the authors use the method of Widom [W], which is based in turn on [TW], together with the asymptotic analysis for orthogonal polynomials in [DKMVZ]. A new and challenging feature of the method in [W], which does not arise in the proof of universality in the case $\beta = 2$, is the appearance of the *inverse* of a certain matrix C_{11} of fixed size $n = 2m - 1$ (see [DG1, (1.37) and Theorem 2.3 et seq.]). In the scaling limit as $N \to \infty$, the matrix C_{11} converges to a matrix C_{11}^{∞} and

$$\det C_{11}^{\infty} = (\det T^{[m-1]})^2 \tag{7}$$

(see discussion from (2.13) up to Theorem 2.4 in [DG1]). Thus in order to control the scaling limit for $\beta = 1$ and 4, we need to show that $\det T^{[m-1]} \neq 0$.

It turns out that $\det T^{[m-1]}$ is related to partition functions for finite log gases in an external field V at inverse temperatures $\beta = 1, 2, 4$

$$\begin{aligned} Z_{V,\beta,k} &\equiv \frac{1}{k!}\int \cdots \int \prod_{1\le i<j\le k} |x_i - x_j|^{\beta} e^{-\sum_{i=1}^{k} V(x_i)}\, dx_1 \cdots dx_k \\ &= \frac{1}{k!}\int \cdots \int e^{-\beta \sum_{1\le i<j\le k} \log|x_i - x_j| - \sum_{i=1}^{k} V(x_i)}\, dx_1 \cdots dx_k. \end{aligned} \tag{8}$$

Using standard formulae for such partition functions (see e.g. [AvM, (4.4), (4.17), (4.20)]), together with [DG1, (2.18)], one finds (see [St, Remark 2.4], [DG1, Remark 1.5]) that for ensembles of (even) size N

$$\det C_{11} = \left(\frac{1}{2^N (N/2)!} \frac{Z_{2V,4,N/2}\, Z_{V,1,N}}{Z_{2V,2,N}} \right)^2. \tag{9}$$

Thus

$$\lim_{N\to\infty} \frac{Z_{2V,4,N/2}\, Z_{V,1,N}}{2^N (N/2)!\, Z_{2V,2,N}} = \det T^{[m-1]} \neq 0. \tag{10}$$

Formula (9), together with (7), raises the possibility of using the methods of statistical mechanics to prove (5), (6). The estimates in [J] show that the partition functions $Z_{V,\beta,k}$ have, for certain constants $\alpha_{V,\beta}$, leading order asymptotics of the form $e^{\alpha_{V,\beta} k^2 (1+o(1))}$ as $k \to \infty$, and moreover, their combined contributions to $\det C_{11}$ cancel to this order. In order to achieve cancellation at subsequent orders, and so prove (5), (6), one needs higher order asymptotics for the $Z_{V,\beta,k}$'s, but, unfortunately such asymptotics are known only for $\beta = 2$ (see [EM]). Regarding (9), we take the contrary point of view, i.e., (10) and (6) provide new information on the asymptotics of partition functions for log gases at inverse temperatures $\beta = 1$ and 4.

Much of the analysis in [DG1] involves estimating $Q(q)$ in two regions: $3 \le q \lesssim \sqrt{m}$ and $\sqrt{m} \lesssim q \le 4m - 5$. In this note, using bounds on

$$W(x) \equiv \frac{2}{\pi} \int_0^x \frac{\sin qs}{\sin s}\, ds \tag{11}$$

which are uniform in $q = 3, 5, \cdots$ and in $0 \le x \le \pi/2$ (see Lemma 4 below), we are able to estimate $Q(q)$ uniformly for $q = 3, 5, \cdots, 4m - 5$ and so avoid many of the

technicalities in the proof in [DG1] of (5). Of course the function $W(x)$ is familiar from the analysis of the Gibbs phenomenon in Fourier analysis.

Remark 1. For $m = 1$, corresponding to the Gaussian orthogonal and symplectic ensembles, $T^{[m-1]}$ is not defined and no analog of (5), (6) is needed (see [DG1]).

We use the following result. For a matrix X let $r(X) = \sup\{|\lambda| : \lambda \in \operatorname{spec} X\}$ denote the spectral radius of X. As is well known, for any operator norm $\|\cdot\|$ on $\{X\}$,

$$r(X) = \lim_{j\to\infty} \|X^j\|^{1/j} = \inf_{j\geq 1} \|X^j\|^{1/j}. \tag{12}$$

Lemma 2. *Assume K and K' are J-dimensional matrices with real entries such that $|K_{ij}| \leq K'_{ij}$, $1 \leq i, j \leq J$, and $r(K') < 1$. Then $r(K) < 1$ and*

$$\det(I - K) \geq \det(I - K') > 0. \tag{13}$$

Proof. The following is true: if $r(X) < 1$, then

$$\det(I - X) = e^{-\sum_{l=1}^{\infty} \frac{1}{l} \operatorname{tr}(X^l)}. \tag{14}$$

This result is usually stated in the form that (14) holds if $\|X\| < 1$ (see e.g. [ReSi]). To obtain (14) for $r(X) < 1$ from the case $\|X\| < 1$ simply apply (14) to μX for μ small and observe that for any fixed ρ satisfying $r(X) < \rho < 1$, $\|X^l\| \leq \rho^l$ for l sufficiently large: then (14) follows for $r(X) < 1$ by analytic continuation $\mu \to 1$.

Equip $\mathbb{R}^J$ with the l_∞-norm $\|\cdot\|_\infty$ (any l_p-norm, $1 \leq p \leq \infty$ would do) and for a matrix X mapping $\mathbb{R}^J \to \mathbb{R}^J$ denote the associated operator norm by $\|X\|$. For $\phi = \{\phi_j\} \in \mathbb{R}^J$ we denote the vector with coordinates $\{|\phi_j|\}$ by $|\phi|$. We claim that $r(K) \leq r(K')$. Indeed, for $\phi \in \mathbb{R}^J$, $|(K^l\phi)_j| \leq ((K')^l|\phi|)_j$ and so

$$\|K^l\phi\|_\infty \leq \|\, (K')^l|\phi|\,\|_\infty \leq \|(K')^l\|\; \|\,|\phi|\,\|_\infty = \|(K')^l\|\, \|\phi\|_\infty.$$

Thus $\|K^l\| \leq \|(K')^l\|$ and so $r(K) \leq r(K') < 1$ by (12). It follows that (14) is valid for K and K'. But clearly $|\operatorname{tr}(K^l)| \leq \operatorname{tr}((K')^l)$ and (13) is now immediate. $\square$

The function $h(x)$ in (1) has the following properties (see [DG1, Proposition 6.2]): for $0 < x < 1$

$$\begin{aligned}
&(i) \quad h \text{ solves the differential equation} \\
&\qquad x(x^2-1)h' + (2m-1-2(m-1)x^2)h = 4m \\
&(ii) \quad \frac{4m}{2m-1} = h(0) \leq h(x) \leq h(1) = 4m \\
&(iii) \quad h(x) = \frac{4mx^{2m-1}}{\sqrt{1-x^2}} \int_x^1 \frac{t^{-2m}}{\sqrt{1-t^2}}\, dt.
\end{aligned} \tag{15}$$

Property (i) reflects the fact that h is a hypergeometric function,

$$h(x) = \frac{4m}{2m-1}\, {}_2F_1(1, -m+1, -m+3/2; x^2)$$

(see [DG1, (6.11)]) and (iii) follows by integrating (i). Property (ii) follows from (i) and (1).

Set

$$u(x) \equiv u(x; m) = \frac{1}{h(x)} - \frac{1-x^2}{2} + \frac{1}{4m}. \tag{16}$$

Note that the function $u(x)$ is closely related to the function y_m which plays a prominent role in [DG1]: we have

$$u(x) = \frac{\sqrt{1-x^2}}{m}\, y_m(\arcsin x) + \frac{1}{2m}, \qquad 0 \le x \le 1.$$

Also note that using the elementary identities for $q = 3, 5, \cdots$,

$$\frac{2}{\pi}\int_0^{\pi/2} \sin qs \, \sin s \, ds = 0, \qquad W\Big(\frac{\pi}{2}\Big) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sin qs}{\sin s}\, ds = 1$$

we have from (2), (16)

$$I(q) = \frac{4}{\pi}\int_0^{\pi/2} \frac{\sin qs}{\sin s}\, u(\cos s)\, ds - \frac{1}{2m}. \tag{17}$$

The main technical result in our proof of Theorem 1 is the following.

Lemma 3. *The function $u(x) = u(x;m)$, $m \ge 2$, has the following properties.*

(i) $u(x)$ is unimodal for $x \in [0,1]$. More precisely, there exists $x_0 \in (0,1)$ such that $u'(x) < 0$ for $0 < x < x_0$ and $u'(x) > 0$ for $x_0 < x < 1$.

(ii) $u(0) = 0$, $u(1) = \frac{1}{2m}$.

(iii) For $0 \le x \le 1$,

$$-\frac{1}{4m} < u(x) \le \frac{1}{2m}.$$

The proof of Lemma 3 is given after the proof of Theorem 1 below. We also need the following elementary result from Fourier analysis.

Lemma 4. *For $q \ge 3$, $0 \le x \le \pi/2$,*

$$0 \le W(x) \le \frac{\sqrt{3}}{\pi} + \frac{2}{3} < 1.218.$$

Proof. As the factor $\sin s$ in $W(x) = \frac{2}{\pi}\int_0^x \frac{\sin qs}{\sin s}\, ds$ is increasing, a standard argument in the analysis of the Gibbs phenomenon shows that for $0 \le x \le \pi/2$, $0 \le W(x) \le \frac{2}{\pi}\int_0^{\pi/q} \frac{\sin qs}{\sin s}\, ds = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{q\sin(t/q)}\, dt$. But for $0 \le t \le \pi/2$, $q \mapsto q\sin(t/q)$ is increasing, and so for $q \ge 3$ and $0 \le x \le \pi/2$, $0 \le W(x) \le \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{3\sin(t/3)}\, dt = \frac{\sqrt{3}}{\pi} + \frac{2}{3}$. □

Assuming Lemma 3, we now prove Theorem 1. By (3), (17), integrating by parts and using Lemma 3(ii),

$$\begin{aligned} Q(q) &= \frac{4}{\pi}\int_0^{\pi/2} \frac{\sin qs}{\sin s}\, u(\cos s)\, ds \\ &= 2\big(W(s)\, u(\cos s)\big)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} W(s)\, u'(\cos s)\, \sin s\, ds \\ &= 2\Big(\int_0^{\arccos x_0} + \int_{\arccos x_0}^{\pi/2}\Big) W(s)\, u'(\cos s)\, \sin s\, ds. \end{aligned}$$

Thus, by Lemma 3 and Lemma 4,

$$\begin{aligned} Q(q) &\le 2 \int_0^{\arccos x_0} W(s)\, u'(\cos s) \sin s \, ds \\ &\le 2 \cdot 1.218 \cdot \big(u(1) - u(x_0)\big) \\ &\le 2 \cdot 1.218 \cdot \Big(\frac{1}{2m} + \frac{1}{4m}\Big) = \frac{3}{2} \cdot \frac{1.218}{m}. \end{aligned}$$

On the other hand

$$\begin{aligned} Q(q) &\ge 2 \int_{\arccos x_0}^{\pi/2} W(s)\, u'(\cos s) \sin s \, ds \\ &\ge 2 \cdot 1.218 \cdot \big(u(x_0) - u(0)\big) \ge -\frac{1}{2} \cdot \frac{1.218}{m} \end{aligned}$$

and thus

$$|Q(q)| \le \frac{3}{2} \cdot \frac{1.218}{m} = \frac{1.827}{m}.$$

Recalling the definitions of $Q^{[m-1]}$ and $B^{[m-1]}$, we have for $1 \le i, j \le m-1$

$$\big|(Q^{[m-1]} B^{[m-1]})_{ij}\big| \le \Big|\sum_{l=1}^{j} Q(n - 2i + 2l)\, 2m \binom{n}{j-l}\Big| \le 2 \cdot 1.827 \cdot \sum_{l=0}^{j-1} \binom{n}{l}$$

and hence

$$\big|(Q^{[m-1]} B^{[m-1]})_{ij}\big| \le 2 \cdot 1.827 \cdot L_{ij}, \qquad 1 \le i, j \le m-1,$$

where L is the rank 1 matrix with entries $L_{ij} = \sum_{l=0}^{j-1} \binom{n}{l}$, independent of i. Hence L has only 1 non-zero eigenvalue $\lambda_1(L)$ and we find

$$\begin{aligned} r(L) = \lambda_1(L) &= \sum_{k=1}^{m-1} L_{1k} = \sum_{k=1}^{m-1} \sum_{l=1}^{k} \binom{n}{l-1} \\ &= \sum_{l=0}^{m-1} (m-l-1) \binom{2m-1}{l} = \frac{m}{2}\binom{2m-1}{m-1} - 2^{2m-3} \le \frac{m}{2}\binom{2m-1}{m-1}. \end{aligned} \tag{18}$$

In the second last step, we have used the elementary formula preceding (6.7) in [DG1].

Assembling the above results and recalling the definition of $K^{[m-1]}$, we obtain for $1 \le i, j \le m-1$,

$$\big|K^{[m-1]}_{ij}\big| = \frac{(m!)^2}{m(2m)!} \big|(Q^{[m-1]} B^{[m-1]})_{ij}\big| \le K'_{ij}$$

where

$$K'_{ij} \equiv 2 \cdot 1.827 \cdot \frac{(m!)^2}{m(2m)!} L_{ij}, \qquad 1 \le i, j \le m-1,$$

and by (18), the only non-zero eigenvalue of K' satisfies

$$\begin{aligned} \lambda_1(K') = r(K') &= 2 \cdot 1.827 \frac{(m!)^2}{m(2m)!} r(L) \\ &\le 2 \cdot 1.827 \frac{(m!)^2}{m(2m)!} \frac{m}{2} \binom{2m-1}{m-1} = \frac{1.827}{2} = 0.9135 < 1. \end{aligned} \tag{19}$$

Thus by Lemma 2,

$$\det(1-K^{[m-1]}) \geq \det(1-K') = 1-\lambda_1(K') \geq 0.0865.$$

This completes the proof of Theorem 1.

Remark 2. Using Lemma 2, the calculations in [DG1] also yield a quantitative version of (5) but with a weaker bound. As above, we estimate $T^{[m-1]}$ elementwise with a *rank one* matrix so that we can estimate the determinant by estimating the only nonzero eigenvalue. We note that we cannot use [DG1, (6.22)] (the matrix in (6.22) is not rank one). For "small" m we use [DG1, (6.15), (6.16)], and for "large" m we use [DG1, (6.55), (6.21)]. We claim that

$$\det T^{[m-1]} \geq 0.02, \qquad m \geq 2. \tag{20}$$

This estimate is not optimal, but we could not strengthen it compared to (6) by the methods in [DG1]. To prove (20) for $2 \leq m \leq 46$, we note that the RHS in [DG1, (6.16)] is < 0.98 for m in this range. (Note that our $Q(q)$ and $\tilde{I}(q)$ in [DG1] are related by $\tilde{I}(q) = mQ(q)-1$ and hence $|Q(q)| = \frac{1}{m}|1+\tilde{I}(q)| \leq \frac{1}{m}(1+|\tilde{I}(q)|)$.) To prove (20) for $m \geq 47$, we set $\delta \equiv 0.04$ and consider $q = 3, 5, \cdots 4m-5$ in the regions $\frac{4}{\pi}\frac{\sqrt{m+1/2}}{q} \leq 1-\delta$ and $\frac{4}{\pi}\frac{\sqrt{m+1/2}}{q} > 1-\delta$ separately. In the former q-region, by [DG1, (6.21)], $|1+\tilde{I}(q)| \leq 1+|\tilde{I}(q)| \leq 1.96$. In the latter q-region, substituting $\frac{q}{\sqrt{m+1/2}} \leq \frac{4}{\pi}\frac{1}{1-\delta}$ in [DG1, (6.55)], we note that the resulting estimate on $|1+\tilde{I}(q)|$ multiplied by $(\frac{1}{2} - \frac{(m!)^2}{m(2m)!}2^{2m-2})$, is < 0.98 in fact for $m \geq 44$. These facts together with Lemma 2 prove (20) (cf. [DG1, (6.56), (6.57)]).

It remains to prove Lemma 3. A straightforward computation using (15)(i) and (16) shows that u is a solution of the equation

$$x(1-x^2)u' - 4mu^2 + (2(m+1)x^2+1-2m)u - \frac{x^2}{2m} = 0. \tag{21}$$

Moreover as $h(x) > 0$, u is smooth. By (15)(ii), and by differentiating (21), we find,

$$\begin{aligned} &u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = -\frac{1}{m(2m-3)} \\ &u(1) = \frac{1}{2m}, \qquad u'(1) = \frac{2}{3} + \frac{1}{3m}. \end{aligned} \tag{22}$$

Now observe that at a point $0 < x < 1$ where $u'(x) = 0$, we cannot have $4m(m+1)u(x)-1 = 0$, i.e. $u(x) = \frac{1}{4m(m+1)}$. Indeed, substituting these values into (21), we find $-1+(1-2m)(m+1) = 0$, which is a contradiction. Next we show that

$$\big(u'(x) = 0 \text{ for some } 0 < x < 1\big) \quad \Longrightarrow \quad u''(x) = \frac{(4m(m+1)u(x)-1)^2}{m(1-2mu(x))(1-4mu(x))}. \tag{23}$$

Indeed, differentiating (21), we find for such a point x

$$u''(x) = \frac{1-4m(m+1)u(x)}{m(1-x^2)}. \tag{24}$$

Setting $u'(x) = 0$ in (21) and solving for $(1-x^2)$ in terms of $u(x)$, we obtain

$$1-x^2 = -\frac{(1-2mu(x))(1-4mu(x))}{4m(m+1)u(x)-1}. \tag{25}$$

Note that by the above argument, the denominator in (25) is non-zero: also the numerator is non-zero as $1 - x^2 \neq 0$. Substituting (25) into (24) we obtain (23). Furthermore, the calculation shows that if $u'(x) = 0$ for some $0 < x < 1$, then $u''(x)$ is (finite and) non-zero.

From (22) we see that for small $x > 0$, $u(x) < 0$. As $u(1) > 0$, there must be at least one point $x \in (0,1)$ where $u(x) = 0$. But it follows from (21) that if $u(x) = 0$, $x \in (0,1)$, then $u'(x) = \frac{x}{2m(1-x^2)} > 0$. Hence u crosses the level zero at a unique point $x_1 \in (0,1)$. Next suppose that $u'(\hat{x}) = 0$ for some $\hat{x} \in (0, x_1)$. But then by (23), $u''(\hat{x}) > 0$ as $u(\hat{x}) < 0$. Thus any critical point for $u(x)$ in $(0, x_1)$, must be a local minimum. As $u(x)$ clearly has a minimum on $(0, x_1)$, it follows that it has a unique minimum at $x_0 \in (0, x_1)$, say, and no other critical points on $(0, x_1)$. Thus $u'(x) < 0$ for $0 < x < x_0$, and $u'(x) > 0$ for $x_0 < x \leq x_1$.

Next we show that

$$0 < u(x) < \frac{1}{2m} \qquad \text{for } x_1 < x < 1. \tag{26}$$

Indeed, if $u(x) = \frac{1}{2m}$ for $0 < x < 1$, then from (21) we find $u'(x) = \frac{2m+1}{2mx} > 0$. But we know from (22) that $u(1) = \frac{1}{2m}$, $u'(1) > 0$. Hence $u(x)$ cannot cross the level $\frac{1}{2m}$ for $0 < x < 1$. This proves (26).

To complete the proof that u is unimodal we show that $u'(x) > 0$ for $x_1 < x < 1$. Suppose $u'(x_2) < 0$ for some $x_1 < x_2 < 1$. Then as $u(x_1) = 0$ and $u(x_2) < u(1) = \frac{1}{2m}$, there must exist $x_1 < x_2^- < x_2$ and $x_2 < x_2^+ < 1$ such that u has a local maximum at x_2^- and a local minimum at x_2^+. By (23), we must have $u(x_2^-) > \frac{1}{4m}$ and $u(x_2^+) < \frac{1}{4m}$. This implies, in particular, that $u(x)$ crosses the level $\frac{1}{4m}$ at at least one point $x^\# \in (x_2^-, x_2^+)$ such that $u'(x^\#) \leq 0$. But by (21), $u(x) = \frac{1}{4m}$, $0 < x < 1$, implies $u'(x) = \frac{1}{2x} > 0$, which is a contradiction. Thus $u'(x) \geq 0$ on $(x_1, 1)$. On the other hand if $u'(x_3) = 0$ for some $x_1 < x_3 < 1$, then by (23), $u''(x_3) \neq 0$ and so $u'(x)$ changes sign in a neighborhood of x_3, contradicting $u'(x) \geq 0$ on $(x_1, 1)$. Thus $u'(x) > 0$ for all $x_1 \leq x \leq 1$. This completes, in particular, the proof of part (i) of Lemma 3.

It remains to show that $u(x) = u(x; m) > -\frac{1}{4m}$ for $m \geq 2$, $x \in [0,1]$. It turns out that $x = x_m \equiv \sqrt{\frac{m-1}{m+2}}$ plays a distinguished role. More precisely, as we now show,

$$u(x_m) > -\frac{1}{4m} \quad \Longrightarrow \quad \Big(u(x) > -\frac{1}{4m} \text{ for all } x \in [0,1]\Big). \tag{27}$$

To see this, suppose $u(x) = -\frac{1}{4m}$ for some $x \in (0,1)$: then from (21) we obtain

$$u'(x) = \frac{(m+2)x^2 - (m-1)}{2mx(1-x^2)}. \tag{28}$$

Suppose $u(x_m) > -\frac{1}{4m}$. If $u(\hat{x}) \leq -\frac{1}{4m}$ for some $0 < \hat{x} < x_m$, then clearly $u(x^\#) = -\frac{1}{4m}$, $u'(x^\#) \geq 0$ for some point $x^\# \in [\hat{x}, x_m)$. But by (28), $u'(x^\#) < 0$, which is a contradiction. Similarly if $u(\hat{x}) \leq -\frac{1}{4m}$ for some $x_m < \hat{x} < 1$, there must exist a point $x^\# \in (x_m, \hat{x}]$ such that $u(x^\#) = -\frac{1}{4m}$, $u'(x^\#) \leq 0$. But this contradicts (28) as above. This proves (27).

To complete the proof of Lemma 3, we must prove $u(x_m) \equiv u(x_m; m) > -\frac{1}{4m}$, $m \geq 2$. Set $s = 1 - x$. From (15)(iii), we obtain

$$h(x) = \frac{4m(1-s)^{2m-1}}{\sqrt{s(2-s)}} \int_0^s \frac{(1-\tau)^{-2m}}{\sqrt{\tau(2-\tau)}}\, d\tau \leq \frac{4m(1-s)^{2m}}{(2-s)(1-s)\sqrt{s}} \int_0^s \frac{(1-\tau)^{-2m}}{\sqrt{\tau}}\, d\tau.$$

Using the elementary inequality $\frac{1-s}{1-\tau} \leq e^{\tau - s}$ for $0 \leq \tau \leq s \leq 1$, we find

$$h(x) \leq \frac{4me^{-2ms}}{(2-s)(1-s)\sqrt{s}} \int_0^s \frac{e^{2m\tau}}{\sqrt{\tau}}\, d\tau = \frac{4me^{-\mu^2}}{(1-\frac{\mu^2}{4m})(1-\frac{\mu^2}{2m})\mu} \int_0^\mu e^{\lambda^2}\, d\lambda$$

where

$$\mu = \sqrt{2ms} = \sqrt{2m(1-x)}.$$

In order to prove $u(x_m) > -\frac{1}{4m}$, $m \geq 2$, we see that it is sufficient to show that

$$\frac{(1-\frac{\mu^2}{2m})\mu e^{\mu^2}}{2\int_0^\mu e^{\lambda^2}\, d\lambda} - \mu^2 + \frac{1}{1-\frac{\mu^2}{4m}} > 0 \qquad \text{for } \mu = \mu_m = \sqrt{2m(1-x_m)}.$$

By the inequality $(1-\frac{\mu^2}{4m})^{-1} > 1+\frac{\mu^2}{4m}$, and the elementary fact that $1 < \mu_m < \sqrt{3}$, $m \geq 2$, we see that it is sufficient to show

$$F(\mu) \geq \frac{1}{m}\, G(\mu) \qquad \text{for } 1 \leq \mu \leq \sqrt{3}$$

where

$$F(\mu) \equiv \mu e^{\mu^2} + 2(1-\mu^2)\int_0^\mu e^{\lambda^2}\, d\lambda, \qquad G(\mu) \equiv \frac{\mu^2}{2}\Big(\mu e^{\mu^2} - \int_0^\mu e^{\lambda^2}\, d\lambda\Big).$$

But $G(\mu)$ is clearly increasing and so it is enough to show

$$F(\mu) \geq \frac{G(\sqrt{3})}{m} \qquad \text{for } 1 \leq \mu \leq \sqrt{3}. \tag{29}$$

Differentiating $F(\mu)$ we find

$$\begin{aligned} &F(1) = e, \qquad F'(1) = 3e - 4\int_0^1 e^{\lambda^2}\, d\lambda > 2.304 > 0 \\ &F''(1) = 2e - 4\int_0^1 e^{\lambda^2}\, d\lambda > -0.415 \\ &F'''(\mu) \geq 0 \qquad \text{for } \mu \geq 1. \end{aligned}$$

Thus for $1 \leq \mu \leq \sqrt{3}$

$$F(\mu) \geq F(1) + F'(1)(\mu-1) + \frac{F''(1)}{2}(\mu-1)^2 \geq e - \frac{0.415}{2}(\sqrt{3}-1)^2 > 2.607.$$

On the other hand $G(\sqrt{3}) < 41.3$, and if we choose m so that $2.607 > \frac{41.3}{m}$, then (29) will hold. Clearly $m \geq 16$ satisfies this inequality. We conclude that $u(x_m) > -\frac{1}{4m}$ for $m \geq 16$. On the other hand, using Maple (only sums and products are involved), we find from (1), (16)

$$\min_{2 \leq m \leq 15} \Big(u(x_m) + \frac{1}{4m}\Big) > 0.0129 > 0.$$

This completes the proof of Lemma 3, and hence Theorem 1.

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